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INTERSECTION THEOREMS WITH GEOMETRIC CONSEQUENCES

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In this paper we prove that if $\mathcal{F}$ is a family of k-subsets of an n-set, $\mu_0, \mu_1, \dots, \mu_s$ are distinct residues mod p (p is a prime) such that $k \equiv \mu_0 \pmod p$ and for $F \neq F' \in \mathcal{F}$ we have $|F \cap F'| \equiv \mu_i \pmod p$ for some i, $1 \leqq i \leqq s$, then $|\mathcal{F}| \leqq \binom{n}{s}$.

As a consequence we show that if $\mathbf{R}^n$ is covered by m sets with $m < (1+o(1))(1.2)^n$ then there is one set within which all the distances are realised.

It is left open whether the same conclusion holds for composite p.

1. Introduction

Let $\mathcal{F}$ be a family of k-element subsets of $\{1, 2, \dots, n\}$, and suppose that $L = \{l_1, l_2, \dots, l_s\}$ is a subset of $\{0, 1, \dots, k-1\}$.

Let us further suppose that for $F, F' \in \mathcal{F}$ we have

(1) $$|F \cap F'| \in L.$$

Ray-Chaudhuri and Wilson [18] proved that (1) implies

(2) $$|\mathcal{F}| \leqq \binom{n}{s}.$$

Deza, Erdős and Frankl [2] proved that for $n > n_0(k)$, (2) can be improved to

(3) $$|\mathcal{F}| \leqq \prod_{i=1}^{s} \frac{n-l_i}{k-l_i}.$$

In this paper we prove

Theorem 1. *Suppose* $\mu_0, \mu_1, \dots, \mu_s$ *are distinct residues modulo a prime* p, *such that*

(4) $$|F| = k \equiv \mu_0 \pmod p,$$

AMS subject classification (1980): 05 C 65; 05 C 35, 05 C 15 *and for any two distinct* $F, F' \in \mathcal{F}$

(5) $$|F \cap F'| \equiv \mu_i \pmod p \quad \textit{for some } i, \quad 1 \leqq i \leqq s.$$

Then

(6) $$|\mathcal{F}| \leqq \binom{n}{s}.$$

Clearly Theorem 1 generalizes (2). It would be interesting to know whether it holds for composite p as well. In this direction, we prove only

Theorem 2. *Let q be a prime power. Suppose that for* $F, F' \in \mathcal{F}$ *we have*

(7) $$|F \cap F'| \not\equiv k \pmod q.$$

Then

(8) $$|\mathcal{F}| \leqq \binom{n}{q-1}.$$

Let $\mathbf{R}^n$ denote n-dimensional Euclidean space. Let us construct a graph on $\mathbf{R}$ by connecting two points if and only if their distance is 1. Let $c(\mathbf{R}^n)$ denote the chromatic number of this graph. The exact value of $c(\mathbf{R}^n)$ seems to be hard to determine. It is known that $4 \leqq c(\mathbf{R}^2) \leqq 7$. Erdős conjectured that $c(\mathbf{R}^n)$ is exponential in n. We prove this conjecture in

Theorem 3.

(9) $$c(\mathbf{R}^n) \geqq (1+o(1))(1.2)^n.$$

Let $m(n)$ be the minimum integer m such that $\mathbf{R}^n$ can be partitioned into m sets $X_1, \ldots, X_m$ such that for $1 \leqq i \leqq m$, there is a real number r_i with the property that $d(x, y) \neq r_i$ for all $x, y \in X_i$ $\big(d(x, y)$ denotes the Euclidean distance, i.e., the length of $x-y\big)$.

This problem was first considered by Hadwiger [13, 14] in 1944 and 1945. Raiskii [17] proved $m(n) \geqq n+2$. This bound was improved by Larman, Rogers [16], then by Larman [15], and again later by Frankl [8]. However none of the lower bounds is exponential. Larman, Rogers [16] proved that

(10) $$m(n) \leqq (3+o(1))^n,$$

and they conjectured that $m(n)$ is exponential in n. Here we prove this conjecture.

Theorem 4.

(11) $$m(n) \geqq (1+o(1))(1.2)^n.$$

The statement of Theorem 4 will follow from the proof of Theorem 3 using Theorem 2 of Larman, Rogers [16] which states the following:

If s is a set of M points in $\mathbf{R}^n$ with critical distance 1 and critical number D (i.e., every subset of s of cardinality exceeding D contains 2 points at distance 1), then

(12) $$m(n) \geqq M/D.$$

We prove as well a modification (Conjecture 2 of Larman, Rogers [16]):

Theorem 5. *Let T be a set of m vectors in $\mathbf{R}^n$*

$$\mathbf{y}^{(i)} = (y_1^{(i)}, y_2^{(i)}, \ldots, y_n^{(i)}); \quad i = 1, \ldots, m,$$

with

$$y_j^{(i)} = \pm 1, \quad i = 1, \ldots, m;$$

$y_j^{(i)} = \pm 1$ *for* $\frac{n}{2}$ *values of* $1 \leqq j \leqq n$, *such that none of the scalar products* $\langle \mathbf{y}^{(i)}, \mathbf{y}^{(j)} \rangle$ *is zero. Then for* $n = 4p^{\alpha}$ *(p prime, $\alpha \geqq 1$) we have*

$$m \leqq 2\binom{n-1}{\frac{n}{4}-1} \leqq (1+o(1))2^n/(1.13)^n. \tag{13}$$

Let B denote the boundary of the unit sphere in $\mathbf{R}^n$ centered at the origin. Let E be a measurable subset of B. H. S. Witsenhausen asked for the value of the supremum of the ratio of the measures of E and B, assuming that E does not contain two points A_1, A_2 which subtend an angle of 90° with the center of the sphere. Let $s(n)$ denote this supremum. Choosing $E_0 = \{\mathbf{y} \in B\colon y_i > 0,\ i = 1, \ldots, n\} \cup \cup \{\mathbf{y} \in B\colon y_i < 0,\ i = 1, \ldots, n\}$ we see that

$$s(n) \geqq 2^{-n+1}. \tag{14}$$

We prove

Theorem 6.

$$s(n) \leqq (1+o(1))(1.13)^{-n}. \tag{15}$$

For $n > k > l \geqq 0$, let $m(n, k, l)$ denote the maximum number of k-subsets of an n-set such that no two of them intersect in l-elements. Erdős [5] conjectured that for $n \geqq n_0(k)$, $k \geqq 4$, we have

$$m(n, k, l) \leqq \max\left\{\binom{n-l-1}{k-l-1}, \binom{n}{l}\Big/\binom{k}{l}\right\}. \tag{16}$$

Here $\binom{n-l-1}{k-l-1}$ corresponds to all the k-subsets containing a fixed $(l+1)$-set while $\binom{n}{l}\Big/\binom{k}{l}$ would correspond to a (n, k, l)-Steiner system. In the first case all the intersections have cardinality greater than l, in the second smaller than l.

Frankl [8] proved that for $k \geqq 3l+2$

$$m(n, k, l) \leqq (1+o(1))\binom{n-l-1}{k-l-1}. \tag{17}$$

Here we prove

Theorem 7. *If $k-l$ is a power of a prime and*

(*a*) $k \geqq 2l+1$, *then*

$$m(n, k, l) = (1+o(1))\binom{n-l-1}{k-l-1}; \tag{18}$$

(b) $k<2l+1$, *setting* $d=2l-k+1$ *we have*

$$m(n,k,l) \leqq \frac{\binom{n}{d}}{\binom{k}{d}}\binom{n-d}{l-d} = O\left(\binom{n}{l}\right). \tag{19}$$

Let $r(k)$ denote the minimum n such that every graph on n vertices contains either a complete or an empty subgraph on k vertices. Erdős [6] proved

$$r(k) > 2^{k/2}. \tag{20}$$

His proof is probabilistic and in [7] he asked for a constructive bound yielding $r(k)>k^t$ for every t for $k>k_0(t)$. Such a construction was given in [9].

Here we use Theorem 2 to give a more accurate construction, though still far from the bound (20) (see Theorem 8).

Let $f(n,k,2)$ denote the maximum cardinality of a collection of $\binom{k}{2}$-subsets of an $\binom{n}{2}$-set such that all the pairwise intersections have for cardinality $\binom{i}{2}$ for $i=1,2,\dots,k-1$.

For $F \subseteqq \{1,2,\dots,n\}$ set $F(2)=\{\{x,y\}\colon x\neq y,\ x,y\in F\}$,

$$\mathcal{G} = \{F(2)\colon F \subseteqq \{1,2,\dots,n\},\ |F|=k\}.$$

Then $\mathcal{G}$ shows that

$$f(n,k,2) \geqq \binom{n}{k}. \tag{21}$$

Frankl [10] conjectured that for $n>n_0(k)$, $k\geqq 10$ we have equality in (21). Here we prove

Theorem 9. *If p is an odd prime then we have*

$$f(n,p,2) \leqq \frac{\binom{n}{2}}{\binom{p}{2}}\binom{\binom{n}{2}}{\frac{p-1}{2}}. \tag{22}$$

In [11] it is conjectured that if $\mathcal{F}$ is a collection of 7-element subsets of an n-set such that all the pairwise intersections have cardinality 0, 2, 3, 5 or 6 then $|\mathcal{F}|=O(n^2)$. We prove

Theorem 10. *Let $\mathcal{F}$ be a collection of 7-subsets of an n-set, such that for $F, F'\in\mathcal{F}$ we have*

$$|F\cap F'|\in\{0,2,3,5,6\}.$$

Then

$$|\mathcal{F}| < \binom{n}{2}.$$

In the last paragraph we mention some possible extensions of Theorem 1. In particular we prove:

Theorem 11. *Suppose* $0 \leqq l_1 < l_2 < \ldots < l_s < n$ *are integers and* $\mathcal{F}$ *is a collection of subsets of* $\{1, 2, \ldots, n\}$ *such that for* $F \neq F' \in \mathcal{F}$ *we have*

$$|F \cap F'| \in \{l_1, \ldots, l_s\}.$$

Then

$$|\mathcal{F}| \leqq \sum_{i=0}^{s} \binom{n}{i}.$$

Note that we do not assume anything about $|F|$.

2. The proof of Theorem 1

Let $A_1, A_2, \ldots, A_{\binom{n}{j}}$ be all the j-subsets and $B_1, B_2, \ldots, B_{\binom{n}{i}}$ be all the i-subsets of $\{1, 2, \ldots, n\}$ with $j > i$.

Let us define the $\binom{n}{i}$ by $\binom{n}{j}$ matrix $N(i, j)$ in the following way: the (u, v)-entry is 1 if $B_u \subset A_v$ and 0 if $B_u \not\subset A_v$ for $1 \leqq u \leqq \binom{n}{i}$, $1 \leqq v \leqq \binom{n}{j}$.

For $i = s$, $j = k$ let the row-vectors be $v_1, v_2, \ldots, v_{\binom{n}{s}}$. They are all vectors in $\mathbf{R}^{\binom{n}{k}}$. Let V denote the vector space generated by the v_i's, $1 \leqq i \leqq \binom{n}{s}$. Obviously we have

$$\dim V \leqq \binom{n}{s}. \tag{23}$$

The following identity can be checked easily $(0 \leqq i < s)$

$$N(i, s) N(s, k) = \binom{k-i}{s-i} N(i, k). \tag{24}$$

Consequently, for $0 \leqq i < s$, the row vectors of $N(i, k)$ are contained in V.

Let us count the product $N(i, k)^T N(i, k) = M(i, k)$, where N^T denotes the transpose of N. Of course $M(i, k)$ is an $\binom{n}{k}$ by $\binom{n}{k}$ matrix in which the $(u\ v)$, entry is $\binom{|A_u \cap A_v|}{i}$ for $1 \leqq u, v \leqq \binom{n}{k}$. Moreover the row-vectors of $M(i, k)$ are linear combinations of the rows of $N(i, k)$, and consequently they are contained in V.

Let us choose $0 \leqq a_i < p$ for $0 \leqq i \leqq s_0$ in such a way that for every integer x we have

$$\prod_{i=1}^{s} (x - \mu_i) \equiv \sum_{i=1}^{s} a_i \binom{x}{i} \quad \text{(modulo } p). \tag{25}$$

Let us set $M=\sum_{i=1}^{s} a_i M(i, k)$, where the addition is to be done componentwise, i.e., in position (u, v) of M we have

(26) $$M(u, v) = \sum_{i=1}^{s} a_i \binom{|A_u \cap A_v|}{i}.$$

By the definition of M the row-vectors of M are in V, and consequently (23) gives:

(27) $$\operatorname{rank} M \leqq \dim V \leqq \binom{n}{s}.$$

Now let $M(\mathcal{F})$ be the minor spanned by the elements $m(u, v)$ for which $A_u, A_v \in \mathcal{F}$.

The assumptions of the theorem and (25) and (26) yield that for $A_u, A_v \in \mathcal{F}$, $u \neq v$, we have

$$m(u, v) \equiv 0 \pmod p$$

and

$$m(u, u) \not\equiv 0 \pmod p.$$

Consequently the determinant of $M(\mathcal{F})$ is not congruent to 0 modulo p, whence $\det M(\mathcal{F}) \neq 0$. Thus using (27) we infer

$$|\mathcal{F}| = \operatorname{rank} M(\mathcal{F}) \leqq \operatorname{rank} M \leqq \binom{n}{s}. \quad \blacksquare$$

Now we prove Theorem 2. We need an easy lemma.

Lemma. *Let* $q=p^\alpha$, p *is a prime,* $\alpha \geqq 1$. *Then for* $a \geqq 0$ $p \Big| \binom{a}{q-1}$ *if and only if* $a \not\equiv -1 \pmod q$.

The proof of the lemma is elementary and we leave it to the reader.

Let us choose real numbers a_i, $0 \leqq i < q$, such that

$$\sum_{i=0}^{q-1} a_i \binom{x}{i} = \binom{x-k-1}{q-1}.$$

Then by the lemma all the off-diagonal entries are zero mod p in the minor corresponding to $\mathcal{F}$ of the matrix $M=\sum_{i=0}^{q-1} a_i M(i, k)$, but the diagonal entries are nonzero mod p consequently the minor is again of full rank, yielding

$$|\mathcal{F}| \leqq \operatorname{rank} M \leqq \binom{n}{q-1}. \quad \blacksquare$$

3. The proof of Theorems 3 and 4

Let us consider the set S of vectors $\mathbf{x}=(x_1, \dots, x_n)$ in $\mathbf{R}^n$ for which $x_i=0$ $(n-2q+1)$-times and $x_i=1/\sqrt{2q}$ the remaining $(2q-1)$ times. Then

$$|S| = \binom{n}{2q-1}.$$

Let us associate with $\mathbf{v}\in S$ the $(2q-1)$-set $F(\mathbf{v})=\{i\colon x_i\neq 0\}$. Then obviously $d(\mathbf{x},\mathbf{y})=1$ is equivalent to $|F(\mathbf{x})\cap F(\mathbf{y})|=q-1$. Thus by Theorem 2 among any $\binom{n}{q-1}+1$ vectors in S there are two at distance 1, i.e., every color contains at most $\binom{n}{q-1}$ of them, yielding

$$c(\mathbf{R}^n) \geqq \max_{q \text{ is a prime power}} \binom{n}{2q-1}\Big/\binom{n}{q-1}.$$

Now choosing q to be $(1+o(1))\dfrac{2-\sqrt{2}}{2}n$ we obtain

$$c(\mathbf{R}^n) \geqq (1+o(1))(1.2)^n.$$

Remark. Since for $q=2^{2l+1}$ the expression $1/\sqrt{2q}=2^{-l-1}$ is rational, the same method yields that the chromatic number of the set of vectors with rational coordinates is exponential as well.

The statement of Theorem 4 follows now from the fact that the set S has critical distance 1 and critical number $\binom{n}{q-1}$ (cf. the introduction).

4. The proof of Theorem 7

(*a*) Since $k\geqq 2l+1$ then $k-l>l$. Thus l is the only integer between 0 and $k-1$ which is congruent to $k \pmod q = k \pmod{(k-l)}$. We can apply Theorem 2, and obtain

$$m(n,k,l) \leqq \binom{n}{k-l-1} = (1+o(1))\binom{n-l-1}{k-l-1},$$

proving (18).

(*b*) For a d-subset D of $\{1, 2, \dots, n\}$ let $\mathscr{G}(D)$ be the collection of those members of the family which contain D. Of course

$$\sum_D |\mathscr{G}(D)| = m\binom{k}{d}.$$

Hence we can choose D_0 such that

(28) $$|\mathscr{G}(D_0)| \geqq m\binom{k}{d}\Big/\binom{n}{d}.$$

Set $\mathcal{F}=\{G-D_0\colon G\in\mathcal{G}(D_0)\}$. Then $\mathcal{F}$ is a family of $(k-d)$-subsets of the $(n-d)$-set $\{1, 2, \dots, n\}-D$, no two of which intersect in $l-d$ elements. Since $k-l>l-d$ we can apply Theorem 2, which gives

$$(29)\qquad |\mathcal{F}| \leqq \binom{n-d}{k-l-1} = \binom{n-d}{l-d}.$$

From (28) and (29) we obtain

$$m(n, k, l) \leqq \binom{n}{d}\Big/\binom{k}{d}\binom{n-d}{l-d} = O\left(\binom{n}{l}\right). \ \blacksquare$$

5. The proof of Theorem 5 and Theorem 6

Let us define $F_i=\{j\colon y_j^{(i)}=+1\}$. Then $|F_i|=2p^\alpha$, and the condition implies $|F_i\cap F_{i'}|\neq p^\alpha$.

Now apply Theorem 7 with $k=2p^\alpha$, $l=p^\alpha$, $d=1$, and deduce

$$m \leqq 2\binom{4p^\alpha-1}{p^\alpha-1} \leqq (1+o(1))2^n/(1.13)^n. \ \blacksquare$$

To prove Theorem 6 we choose q to be the smallest prime power which is at least $n/4$. Let α, β be two real numbers and let $S(\alpha,\beta)$ be the set of vectors $\mathbf{y}=(y_1, y_2, \dots, y_n)$ for which

$$y_i=\alpha \quad (2q-1) \text{ times,} \quad \text{and} \quad y_i=\beta \quad (n-2q+1) \text{ times.}$$

For $\mathbf{y}\in S(\alpha,\beta)$ set $F(\mathbf{y})=\{i\colon y_i=\alpha\}$. Now the length of $\mathbf{y}$ is $\sqrt{(2q-1)\alpha^2+(n-2q+1)\beta^2}$, i.e., $\mathbf{y}$ is on B iff

$$(30)\qquad (2q-1)\alpha^2+(n-2q+1)\beta^2 = 1.$$

If $|F(\mathbf{y})\cap F(\mathbf{y}')|=q-1$ then

$$\langle\mathbf{y}, \mathbf{y}'\rangle = (q-1)\alpha^2+(n-3q+1)\beta^2+2q\alpha\beta.$$

To make this scalar product vanish we need

$$(31)\qquad (q-1)\alpha^2+(n-3q+1)\beta^2+2q\alpha\beta = 0.$$

Since $q\geqq \frac{n}{4}$ the system (30), (31) is solvable in real α, β. Let S be the image of $S(\alpha,\beta)$ under any orthogonal transformation of B. Then $|S|=|S(\alpha,\beta)|=\binom{n}{2q-1}$, and applying Theorem 2 with $k=2q-1$, the special choice above of α,β gives:

$$(32)\qquad \frac{|E\cap S|}{|B\cap S|} = \frac{|E\cap S|}{|S|} \leqq \frac{\binom{n}{q-1}}{\binom{n}{2q-1}} \leqq (1+o(1))(1.13)^{-n}.$$

Now averaging over the orthogonal group yields

$$\frac{\mu(E)}{\mu(B)} \leqq \max_S \frac{|E\cap S|}{|S|} \leqq (1+o(1))(1.13)^{-n},$$

yielding (15). ▮

6. Constructive Ramsey-bound

Theorem 8. *Let us set* $V(\mathcal{G})=\{F\subseteq\{1,2,\dots,n\}\colon |F|=q^2-1\}$, *$q$ is a prime power, and* $E(\mathcal{G})=\{(F,F')\colon |F\cap F'|\not\equiv -1 \pmod q\}$.

Then $\mathcal{G}$ contains no complete or empty subgraph on more than $\binom{n}{q-1}$ vertices.

Proof. If $F_1,\dots,F_m$ is a complete subgraph then $|F_i\cap F_j|\not\equiv -1 \pmod q$ for every $1\leqq i<j\leqq m$. Thus Theorem 2 gives the assertion.

If $F_1,\dots,F_m$ is an empty subgraph then $|F_i\cap F_j|\in\{q-1, 2q-1, \dots, q^2-q-1\}$ for $1\leqq i<j\leqq m$, thus (2) gives the statement. ▮

Setting $n=p^3$, $q=p$, we obtain

$$r(k)\geqq \exp\left((1+o(1))\log^2 k/4\log\log k\right).$$

7. The proof of Theorems 9 and 10

Let x be a point of maximal degree and set

$$\mathcal{F}_0=\{F\in\mathcal{F}\colon x\in F\}.$$

Then

(33) $$|\mathcal{F}_0|\geqq|\mathcal{F}|\binom{p}{2}\Big/\binom{n}{2},$$

and for $F, F'\in\mathcal{F}_0$ we have

$$|F\cap F'|\in\left\{\binom{2}{2}, \binom{3}{2}, \dots, \binom{p-1}{2}\right\}.$$

Since $\binom{i}{2}-\binom{p-i+1}{2}=\frac{(2i-1)p-p^2}{2}\equiv 0 \pmod p$, and $p\nmid\binom{i}{2}$ for $i=2,\dots,p-1$, the intersections lie in $\frac{p-1}{2}$ different non-zero congruence classes modulo p. On the other hand $p\,\Big|\binom{p}{2}=|F|$, and therefore Theorem 1 yields

(34) $$|\mathcal{F}_0|<\left(\begin{matrix}\binom{n}{2}\\ \binom{p-1}{2}\end{matrix}\right).$$

Now (33) and (34) imply (22). ▮

Theorem 10 is an immediate consequence of Theorem 1: Simply set $k=7$, $\mu_0=1$, $\mu_1=0$, $\mu_2=2$, $p=3$. ▮

8. On possible extensions

First we prove Theorem 11.

Let $F_1, F_2, \ldots, F_m$ be the sets in our family arranged so that $|F_1| \geqq |F_2| \geqq \geqq \ldots \geqq |F_m|$.

For $0 \leqq i \leqq s$, let $A_1, \ldots, A_{\binom{n}{i}}$ be the different i-subsets of $\{1, 2, \ldots, n\}$.

Let $N(i)$ be the m by $\binom{n}{i}$ matrix which has 1 or 0 in the position (u, v) according to whether $A_v \subset F_u$ or not, $1 \leqq u \leqq m$, $1 \leqq v \leqq \binom{n}{i}$. Of course $r(N(i)) \leqq \binom{n}{i}$. Let us set $M(i) = N(i)N(i)^T$. Then $M(i)$ is m by m with $\binom{|F_u \cap F_v|}{i}$ in position (u, v), and we still have

$$r(M(i)) \leqq \binom{n}{i}.$$

Let $v_1^{(i)}, \ldots, v_m^{(i)}$ be the row-vectors of $M(i)$, and let V be the vector space spanned by the $v_j^{(i)}$ for $1 \leqq i \leqq s$, $1 \leqq j \leqq m$. Then we have

$$\dim V \leqq \sum_{i=0}^{s} r(M(i)) \leqq \sum_{i=0}^{s} \binom{n}{i}. \tag{35}$$

Let us choose $a_\nu^{(i)}$ for fixed i, $1 \leqq i \leqq s$, and $\nu = 0, 1, \ldots, i$ that

$$\sum_{\nu=0}^{i} a_\nu^{(i)} \binom{x}{\nu} = \prod_{t=1}^{i} (x - l_t). \tag{36}$$

Now we define an m by m matrix M. If $1 \leqq u \leqq m$ and i is the greatest integer for which $|F_u| > l_i$ then let the uth row of M be

$$\sum_{\nu=0}^{i} a_\nu^{(i)} v_u^{(\nu)}. \tag{37}$$

If $u = m$, and $|F_u| = l_s$, then the last row of M is $v_m^{(0)}$. Since all the row-vectors are in V we have by (35)

$$r(M) \leqq \sum_{i=0}^{s} \binom{n}{i}. \tag{38}$$

By (36) and (37) the u'th diagonal entry of M is

$$\prod_{t=1}^{t} (|F_u| - l_t) \neq 0, \quad \text{since} \quad |F_u| > l_i.$$

Since $|F_u| \geqq |F_v|$ for $u < v$, in this case $|F_u \cap F_v| \in \{l_1, l_2, \ldots, l_i\}$, and consequently by (26) and (37) the (u, v)-entry of M is 0. This means that M is lower-triangular with non-zero diagonal consequently of full rank; thus (38) yields

$$|\mathcal{F}| = m = \operatorname{rank} M \leqq \prod_{i=0}^{s} \binom{n}{i}. \quad \blacksquare$$

The most important extension is to decide whether Theorem 1 or at least Theorem 2 holds for congruences modulo arbitrary positive integers.

Frankl, Rosenberg [12] proved that for $s=1$ Theorem 1 extends to arbitrary integer moduli (which generalizes results by Ryser [19], Deza, Erdős, Singhi [3], Babai, Frankl [1], and Deza, Rosenberg [4]).

The first open case modulo a prime power is for 8: $\mu_0=0$, $\mu_1=1$, $\mu_2=2$, $\mu_3=4$ and $\mu_4=6$.

By the proof of Theorem 1 we can prove

Theorem 12. *Suppose q is a power of the prime p. Let $\mu_0, \mu_1, \ldots, \mu_s$ be distinct residues modulo q. Let $\mathcal{F}$ be a collection of k-subsets of $\{1, 2, \ldots, n\}$, such that for $F \neq F' \in \mathcal{F}$ we have*

$$|F| \equiv \mu_0 \pmod q,$$

$$|F \cap F'| \equiv \mu_i \pmod q \quad \text{for some} \quad 1 \leqq i \leqq s.$$

If there exists a rational polynomial $g(x)$ of degree d such that $p \nmid g(k)$ ($g(k)$ is an integer) but $p | g(x)$ for $x \equiv \mu_i \pmod q$, $i=1, \ldots, s$, then

$$|\mathcal{F}| \leqq \binom{n}{d}.$$

Proof. Choose the rational numbers $a_0, a_1, \ldots, a_d$ in such a way that

$$\sum_{\nu=0}^{d} a_\nu \binom{x}{\nu} = p(x).$$

Then the matrix $M = \sum_{\nu=0}^{d} a_\nu M(\nu, k)$ contains a full-rank minor corresponding to the members of $\mathcal{F}$, yielding

$$|\mathcal{F}| \leqq \operatorname{rank} M \leqq \operatorname{rank} M(d, k) \leqq \binom{n}{d}. \quad ∎$$

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